

On the variation of the Astronomical Unit and the corrections to planetary motion on an expanding locally anisotropic background

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ABSTRACT

In this work are computed analytical solutions for orbital motion on a background described by an Expanding Locally Anisotropic (ELA) metric ansatz. This metric interpolates between the Schwarzschild metric near the central mass and the Robertson-Walker metric describing the expanding cosmological background far from the central mass allowing for a fine-tuneable covariant parameterization of gravitational interactions corrections in between these two asymptotic limits. Assuming a non-varying gravitational constant, $\dot{G} = 0$, it is discussed the variation of the Astronomical Unit (AU) obtained from numerical analysis of the Solar System dynamics, being shown that the corrections to the orbital periods on the Solar System due to the decrease of the Sun's mass by radiation emission plus the General Relativity corrections due to the ELA metric background with respect to Schwarzschild backgrounds can be mapped to the reported yearly increase of the AU through the corrections to Kepler's third law. Based on the value of the heuristic fit to the parameter \dot{AU} corresponding to the more recent ephemerides of the Solar System are derived bounds for the value of a constant parameter α_0 for the ELA metric as well as the maximal corrections to orbital precession and orbital radius variation within this framework. Hence it is shown that employing the ELA metric as a functional covariant parameterization to model gravitational interactions corrections within the Solar system allows to avoid the need for a varying AU and/or varying gravitational constant G .

Keywords: orbital motion, Solar System, Astronomical Unit, local anisotropy, cosmological expansion, variation of fundamental constants

1 Introduction

For numerical analysis of the Solar System dynamics it is chosen as standard of measurement the Astronomical Unit (AU). This standard of measurement fixes the measurement projection for spatial-lengths and, through Kepler's third law, relates the measurement projection for spatial-lengths with the measurement projection for temporal-lengths. Specifically the definition of the AU is stated in IS units as [1–4]

$$AU = \left(G M_{\odot} \left(\frac{T_{1AU,0}}{2\pi} \right)^2 \right)^{\frac{1}{3}} = 149597870700 \text{ m} , \quad (1)$$

where $M_{\odot} = 1.9891 \times 10^{30} \text{ kg}$ is the Sun's mass, $G = 6.67 \times 10^{-11} \text{ kg}^{-1} \text{ m}^3 \text{ s}^{-2}$ is the Gravitational constant, $T_{1AU,0} = 31562889.928 \dots \text{ s}$ is the Keplerian orbital period for a point mass in an elliptic orbit with semi-major axis of value 1 AU . This definition exactly matches Kepler's third law for a planet orbiting the Sun in an elliptic orbit with semi-major axis of length $r_{1,\text{orb}} = 1 \text{ AU}$. This definition is based on the existence of a Keplerian constant of motion, specifically the angular momentum $J_0(r_{1,\text{orb}} = 1 \text{ AU}) = -\sqrt{G M AU (1 - e^2)}$, where e is the orbit eccentricity, hence time and space measurements are related by this classical conservation law stated in the definition (1).

Considering experimental data from optical and range measurements of planetary orbital motion plus range measurement from orbiters and landers it was concluded that a yearly variation of the Astronomical Unit is required to fit the ephemerides in the Solar System. Hence adding the additional parameter \dot{AU} to the dynamical model, the best fit to this parameter was originally estimated to be [2, 3]

$$\left\langle \dot{AU} \right\rangle_{\text{Fit}} = +(15 \pm 4) \text{ cm yr}^{-1} . \quad (2)$$

The value for this heuristic fit does not necessarily correspond to a physical radial distance variation in the Solar System, instead it corresponds to a variation of the standard of measurement considered for numerical analysis of the Solar System (1).

As an alternative to the variation of the AU, it has been suggested that a variation of the product of the Gravitational constant by the Solar mass (GM_{\odot}) would have a similar effect in the planetary dynamics. Specifically the best fit corresponds to the value $(\dot{GM}_{\odot})/(GM_{\odot}) = -(5 \pm 4) \times 10^{-14} \text{ yr}^{-1}$ [4]. We note that the variation of (GM_{\odot}) has the effect of increasing both the orbital radius and the orbital period such that angular momentum is approximately conserved. Specifically the period varies as $\dot{T}_0/T_0 \approx -2(\dot{GM}_{\odot})/(GM_{\odot})$ and the orbital radius as $\dot{AU}/AU \approx -(\dot{GM}_{\odot})/(GM_{\odot})$ [5, 6]. Hence, from Kepler's third law (1), a variation for (GM_{\odot}) can be directly mapped to a non-null value for the parameter \dot{AU} while keeping (\dot{GM}_{\odot}) [4]

$$\left\langle \dot{AU} \right\rangle_{\text{Fit}} = 0.75 \pm 0.60 \text{ cm yr}^{-1} . \quad (3)$$

The discrepancy between both fitted values (2) and (3) is due to the choice of the independent set of parameters considered in the fit. While in [2] both AU and \dot{AU} are fitted, in [4] only (\dot{GM}_{\odot}) is fitted. This choice is justified in [4] by noting that AU and \dot{AU} are correlated by 98.1%. We remark that it is also recognized in [4] that the choice of whether to fit the parameter \dot{GM} or \dot{AU} is a matter of convenience and that both cannot be fitted simultaneously as the obtained value of \dot{AU} does not exceeds the respective formal error. Actually both these choices are interpreted as a varying scaling of the measurement standard (1), hence a choice of the measurement projection (see [7] for further details), being directly mapped into eachother through Kepler's third law.

This discussion is enough to conclude that some sort of unmodelled gravitational interaction seems to be acting in the Solar System which is effectively accounted for by an heuristic rescaling of

the relation between spatial lengths and temporal lengths of the dynamical system model. The exact identification of the origin of such interaction is further diffculted due to the only available solution relying on the heuristic fit of one single isotropic parameter \dot{AU} (or equivalently \dot{GM}) across the Solar System. In this work our main objective is to employ a background described by an expanding locally anisotropic (ELA) metric [8, 9] to analytically model the corrections to orbital motion with respect to orbital motion on Schwarzschild background within the Solar System. In particular, based in analytical orbital solutions for the background described by the ELA metric we will explicitly compute the respective corrections to the orbital period due to two distinct contributions, the Sun's mass decrease corrections and the General Relativity corrections with respect to Schwarzschild backgrounds [10]. It is shown that it is possible to map these corrections to the heuristic variation of the AU, having simultaneously negligible contributions to the remaining orbital parameters. Considering the most reliable estimative for the average value of the parameter \dot{AU} (3) we will compute bounds for the values of the metric functional parameter which, in turn, will allow to estimate the contributions due to the ELA metric background to the orbital period, orbital precession and orbital radius variation of the several planets within the Solar System.

The ansatz for the ELA metric was originally suggested as a possible description of local matter distributions in the expanding universe [11, 12], hence interpolating between the Schwarzschild (SC) metric [13] near the central mass and the Robertson-Walker (RW) metric [14] describing the expanding cosmological background far from the central mass. Hence the ELA metric generalizes the isotropic McVittie metric [15] and the anisotropic metrics considered in [16], having the novelty of maintaining the SC event horizon and maintain as the only space-time singularity the SC mass pole at the origin such that the value of the Schwarzschild mass pole is maintained [8, 17]. This metric is locally anisotropic consistently with astrophysical observations [18, 19] converging at large radius to the RW metric such that global isotropy is maintained. Specifically the line-element for the ELA metric is

$$ds^2 = (1 - U_{\text{SC}}) c^2 dt^2 - r_1^2 (d\theta^2 + \sin^2 \theta d\varphi^2) - \frac{1}{1 - U_{\text{SC}}} \left(dr_1 - H \frac{r_1}{c} (1 - U_{\text{SC}})^{\frac{\alpha}{2} + \frac{1}{2}} c dt \right)^2. \quad (4)$$

where $H = \dot{a}/a$ is the time dependent Hubble rate defined by the rate of change of the universe scale factor a , $U_{\text{SC}} = 2GM/(c^2 r_1)$ is the usual Schwarzschild gravitational potential, G is the Gravitational constant, M is the value of the Schwarzschild mass pole for the central mass being considered and c is the speed of light in vacuum. The exponent α is, generally, a function of the radial coordinate, here we will take the simplified ansatz studied in [8]

$$\alpha(r_1) = (\bar{\alpha}_0 - \alpha_1) + \alpha_1 U_{\text{SC}}(r_1) = (\bar{\alpha}_0 - \alpha_1) + \alpha_1 \frac{2GM}{c^2 r_1}, \quad (5)$$

where $\bar{\alpha}_0$ and α_1 are numerical coefficients. The bound $\bar{\alpha}_0 \geq 3$ ensures that the SC radius $r_{1,\text{SC}} = 2GM/c^2$ is an event horizon and space-time is singularity free at this horizon, while for the bound $\bar{\alpha}_0 > 5$ space-time is asymptotically Ricci flat near the event horizon such that the SC metric is a good approximation in a neighbourhood of the point-like mass M . The coefficient $\alpha_1 < 0$ ensures that the singularity at the origin coincides with the SC mass-pole. In the following analysis we will consider it arbitrarily close to null, $-1 \ll \alpha_1 < 0$, such that outside the event horizon its effects are negligible, hence $\alpha \approx \bar{\alpha}_0$ for planetary orbits. For further details in the derivation of the ELA metric ansatz see [8].

We note that so far no direct physical interpretation for the functional parameter α exists, hence it is at most considered to be a functional parameterization of corrections to gravitational interactions for intermediate spatial-length scales. In particular we may assume that it is varying

over the radial coordinate allowing to phenomenologically fit existing data hoping that the results obtained may shed some light on its physical interpretation (see for instance [20]). In order to fit the constant $\bar{\alpha}_0$ to planetary motion within the Solar System maintaining the asymptotic limits of the ELA metric we are considering the simpler possible ansatz for this constant by considering that near the event horizon of the central mass (the Sun) the exponent is some constant $\alpha_{0,0} \geq 3$ and that far from the event horizon (hence for all planetary orbits) the constant $\bar{\alpha}_0$ is approximately given by a constant α_0 which can be either positive, either negative. Specifically we are considering the simplified ansatz

$$\bar{\alpha}_0 = \begin{cases} \alpha_{0,0} \geq 3 & , \quad r_1 \sim R_{\odot SC} \\ \alpha_0 & , \quad r_1 \gg R_{\odot SC} \end{cases} \quad (6)$$

where $R_{\odot SC} = 2GM_{\odot}/c^2 \approx 2952.22 m$ is the Sun's Schwarzschild radius and M_{\odot} is the Sun's mass. Based in this ansatz we will compute the analytical corrections to two body heliocentric orbital motion within the solar system on the ELA metric background (4) with respect to orbital motion on Schwarzschild backgrounds [21, 22]. More accurate results including the many body gravitational interactions in the Solar System can only be computed by extensive numerical analysis including 10000 known bodies in the solar system [23]. These analysis are usually carried either in the Post-Newtonian formalism, either in the Post-Post-Newtonian formalism which includes both the General Relativity corrections to the classical Newton law of gravitation on Schwarzschild backgrounds, as well as corrections of extended theories of gravity such as Brans-Dicke gravity [24].

When required, for numerical evaluation of the Hubble rate H and the deceleration factor q of today's universe, we are considering the values $H_0 = H|_{t=t_0} = 2.28 \times 10^{-18} s^{-1}$ and $q_0 = -\frac{\ddot{a}}{a} \left(\frac{\dot{a}}{a}\right)^{-2} \Big|_{t=t_0} = -0.582$ [19]. As for the planetary orbital parameters considered for numerical evaluations we are considering the data presented in table 1.

Planet	$r_{1,orb} (\times 10^{11} m)$	e	$m (\times 10^{24} kg)$	$\delta_{orb} (deg)$	$\delta_{per} (deg)$
Mercury	0.579091768	0.20563069	0.3302	7.00487	77.45645
Venus	1.08208926	0.00677323	4.8685	3.39471	131.53298
Earth	1.49597887	0.01671022	5.9736	0	102.94719
Mars	2.27936637	0.09341233	0.64185	1.85061	336.04084
Jupiter	7.78412027	0.04839266	1898.6	1.30530	14.75385
Saturn	14.26725413	0.05415060	568.46	2.48446	92.43194
Uranus	28.70972220	0.04716771	86.832	0.76986	170.96424
Neptune	44.98252911	0.00858587	102.43	1.76917	44.97135
Pluto	59.06376272	0.24880766	0.0125	17.14175	224.06676

Table 1: *Planetary orbits parameters considered: the semi-major axis $r_{1,orb}$, the eccentricity e , the mass, the orbital inclination δ_{orb} and the longitude of perihelion δ_{per} [25].*

This work is organized as follows. In section 2 are computed the analytical solutions for orbital motion on the ELA metric background being derived the General Relativity corrections to orbital precession and orbital period for such backgrounds. This derivation is carried considering a static elliptical orbit approximation. In section 3 are analysed circular orbits on the ELA metric background (4) and computed the orbital radius variation within this approximation. In section 4 are analysed the corrections to Kepler's third law due to the decrease of the Sun's mass and due to the ELA metric background. In particular are derived estimative for the several contributions that allow to match the heuristic variation of the AU and computed the value for the ELA metric

parameter that corresponds to the fitted value of the parameter $\dot{A}U$ (3), as well as the respective corrections to orbital precession and orbital radius variation. In the conclusions we shortly resume and discuss the results obtained in this work.

2 Perturbative Static Elliptical Orbit Solutions

In this section we derive analytical solutions for elliptical orbits on the background described by the ELA metric ansatz (4). In particular we will explicitly compute the General Relativity corrections on such backgrounds to the Keplerian elliptical orbit solution $r_1(\varphi) = 1/u_0(\varphi)$ with

$$u_0(\varphi) = \frac{1 + e \cos(\varphi)}{d} \quad , \quad d = r_{1.\text{orb}}(1 - e^2) \quad , \quad (7)$$

where e is the orbit eccentricity and $r_{1.\text{orb}}$ is the orbital radius. We note that such corrections will generally include the General Relativity corrections on Schwarzschild backgrounds as well as corrections depending on the Hubble rate H .

It is hard, if not impossible, to obtain an exact analytical solution considering the differential equations for a time varying Hubble rate H . The main difficulty is that energy conservation is no-longer given by a constant of motion, instead we have a non-linear second order differential equation on the function t coupled to the differential equation for r_1 . Hence, for technical simplification purposes, we are taking the static orbit approximation by considering a fixed Hubble rate $H = H_0$ corresponding to today's measured value for this rate.

From the metric line-element (4) let us consider the Lagrangian definition [21, 22] to order H_0^2

$$\begin{aligned} \frac{\mathcal{L}}{m} = & \left(1 - U_{\text{SC}} - \left(H_0 \frac{r_1}{c} \right)^2 (1 - U_{\text{SC}})^\alpha \right) \left(c \frac{dt}{d\tau} \right)^2 \\ & + 2H_0 \frac{r_1}{c} (1 - U_{\text{SC}})^{\frac{\alpha_0}{2} - \frac{1}{2}} c \frac{dt}{d\tau} \frac{dr_1}{d\tau} - \frac{1}{1 - U_{\text{SC}}} \left(\frac{dr_1}{d\tau} \right)^2 - r_1^2 \left(\frac{d\varphi}{d\tau} \right)^2 + O(H_0^2 t) \quad . \end{aligned} \quad (8)$$

This Lagrangian is a constant $\mathcal{L}/m = c$ and it is considered that the orbit of the test body is lying in the plane of constant coordinate $\theta = \pi/2$ such that $d\theta = 0$ and $\sin\theta = 1$ [21, 22]. The Lagrangian is independent of the coordinate φ , hence a constant of motion corresponding to angular momentum exists being given by the variational derivation of the Lagrangian with respect to $d\varphi/d\tau$,

$$J = \frac{1}{2m} \frac{\delta \mathcal{L}}{\delta \frac{d\varphi}{d\tau}} = -r_1^2 \frac{d\varphi}{d\tau} \quad . \quad (9)$$

Also, due to the Lagrangian (8) not depending explicitly on the time coordinate, a conserved constant of motion corresponding to energy exists being given by the functional variation of the Lagrangian with respect to $c dt/d\tau$

$$\begin{aligned} \frac{2E_H}{m c} = & \frac{1}{m} \frac{\delta \mathcal{L}}{\delta (c dt/d\tau)} \\ = & 2 \left(1 - U_{\text{SC}} - \left(H_0 \frac{r_1}{c} \right)^2 (1 - U_{\text{SC}})^\alpha \right) \left(c \frac{dt}{d\tau} \right) + 2H_0 \frac{r_1}{c} (1 - U_{\text{SC}})^{\frac{\alpha_0}{2} - \frac{1}{2}} \left(\frac{dr_1}{d\tau} \right) \quad . \end{aligned} \quad (10)$$

This equation can be solved for $c dt/d\tau$ such that replacing the obtained solution in the Lagrangian (8), expressing the derivatives with respect to proper time $dr_1/d\tau$ by the derivatives

with respect to φ , $dr_1/d\tau = dr_1/d\varphi \times d\varphi/d\tau$ and considering the change of variables $u = 1/r$, further differentiating with respect to φ and factoring out an overall factor of $2u'J^2$ (with the primed quantities representing derivation with respect to φ), we obtain the approximate differential equation of order H_0^2 for the function $u(\varphi)$ describing an orbiting test mass in the gravitational field of a point-like central mass M

$$\begin{aligned} u'' + u &= \frac{GM}{J^2} + \frac{3GM}{c^2} u^2 \\ &- \alpha_0 \frac{GM}{c^2} \left(\frac{H_0}{c} \right)^2 \left(1 - \frac{2GM}{c^2} u \right)^{-1+\alpha_0} \\ &- \left(\frac{H_0}{J} \right)^2 \frac{1}{u^3} \left(1 - \frac{2GM}{c^2} u \right)^{-1+\alpha_0} \left(1 - \frac{2GM}{c^2} u + \frac{\alpha_0 GM}{c^2} u \right). \end{aligned} \quad (11)$$

The terms in the first line match the usual terms obtained for Schwarzschild backgrounds and the terms in the second and third lines are the corrections due to the ELA metric background.

We note that, although maintaining the terms of order H_0^2 in the Lagrangian (8) which do not depend explicitly on the time coordinate we have neglected one term containing the factor $H_0^2 t$. Explicitly it is the term $-2q_0 H_0^2 t r_1 (1 - U_{\text{SC}})^{(\alpha_0-1)/2} (dt/d\tau) (dr_1/d\tau)$. Comparing the terms of order H_0^2 in the Lagrangian with this term we conclude that this is a valid approximation as long as the value of the time coordinate is below the following bound

$$t \ll -\frac{1}{q_0} \frac{r_1}{dr_1/d\tau} (1 - U_{\text{SC}})^{\frac{\alpha_0}{2}-\frac{1}{2}} \frac{E_H}{mc^2} \sim 10^{13} \text{ years}. \quad (12)$$

This bound is well above any astrophysical measurement time span and has been obtained by considering the following simplified assumptions, within the solar system, from the experimental upper bounds on the orbital radius variations within the Solar System [7] we consider the estimative for the ratio $|r_1/\dot{r}_1| > 10^{20}$, assume weak gravitational field $U_{\text{SC}} \ll 1$ and values of α_0 for which the approximation $(1 - U_{\text{SC}})^{(\alpha_0-1)/2} \sim 1$ is valid. We note that this approximation will no longer be valid for very large values of the metric exponent, $|\alpha_0| \gg 0$.

Noting that for orbits in the solar system the function u has relatively small values ($0.5 \times 10^{-12} < u < 0.5 \times 10^{-10} m^{-1}$, where $r_{1,\text{orb}}$ is the orbit semi-major axis), with the objective of further simplifying the differential equation (11), we consider a series expansion on u of the terms of order H_0^2 which is equivalent to an expansion on the weak gravitational field. Specifically, for a generic exponent p , the factor $(1 - U_{\text{SC}})^p$ has the following series expansion $(1 - 2GM u/c^2)^p = 1 - p 2GM u/c^2 + p(p-1) (2GM/c^2)^2 u^2/2 - p(p-1)(p-2) (2GM/c^2)^3 u^3/6 + O\left(p^4 (2GM/c^2)^4 u^4\right)$. We note that the full series is strictly convergent independently of the value of the exponent p as long as $u < c^2/(2GM)$, however an approximation to first order on u will only be valid as long as $2pGM/c^2 u < 1$, otherwise it is required to consider higher order terms to attain a valid approximation. Hence the differential equation (11) is, to order u^2 , rewritten as

$$\begin{aligned} u''(\varphi) + A u(\varphi) &\approx \frac{GM}{J^2} B + \frac{3GM}{c^2} C u^2 \\ &- \left(\frac{H_0}{J} \right)^2 \frac{1}{u^3} + \alpha_0 \left(\frac{H_0}{J} \right)^2 \frac{GM}{c^2} \frac{1}{u^2} + O(u^3). \end{aligned} \quad (13)$$

We note that with respect to the General Relativity orbit's equation on Schwarzschild backgrounds, there are the extra multiplicative factors $A = 1 + \delta_A$, $B = 1 + \delta_B$ and $C = 1 + \delta_C$. These

factors differ from unity by the following additive constants

$$\begin{aligned}
\delta_A &= -2(\alpha_0 - 1)\alpha_0 \left(\frac{GM H_0}{c^3} \right)^2 \left(1 + (\alpha_0^2 - 5\alpha_0 + 6) \frac{(GM)^2}{3c^2 J^2} \right), \\
\delta_B &= -\alpha_0 \left(\frac{H_0}{c^2} \right)^2 \left(J^2 + (\alpha_0^2 - 3\alpha_0 + 2) \frac{2(GM)^2}{3c^2} \right), \\
\delta_C &= -\frac{2}{3}\alpha_0 (\alpha_0^2 - 3\alpha_0 + 2) \left(\frac{GM H_0}{c^3} \right)^2 \left(1 + (\alpha_0^2 - 7\alpha_0 + 12) \frac{(GM)^2}{5c^2 J^2} \right).
\end{aligned} \tag{14}$$

So far we have not specify for which values of the coefficient α_0 the static approximation considered is valid. By comparing the leading order terms with the next to leading order terms we conclude that this perturbative equation is valid only for absolute values of the parameter α_0 up to

$$|\alpha_0| < \alpha_{0,\text{max.pert}} \approx \frac{c^2 r_{1,\text{orb}}}{2GM}. \tag{15}$$

Above this value it is either necessary to consider higher order terms of the series expansion or to consider the exact expressions. Nevertheless we remark that, for a fixed positive value of the radial coordinate r_1 , and larger positive values of the parameter $\alpha_0 > \alpha_{0,\text{max.pert}}$ the corrections given by the exact expression due to the ELA metric background will decrease significantly in absolute value becoming, for very large values of the parameter $\alpha_0 \gg \alpha_{0,\text{max.pert}}$, negligible, while for larger negative values of the parameter $\alpha_0 < -\alpha_{0,\text{max.pert}}$ the corrections become more significant being unbounded from below. Hence, although for positive values of α_0 the approximation (13) subject to the bound (15) allows to establish a fairly good estimative for the maximum contribution of the corrections on the ELA metric background, for negative values of α_0 no bounds can be set for such contribution. In figure 1 are plotted the values of the exact and perturbative correction terms of order H_0^2 in the differential equation (11).

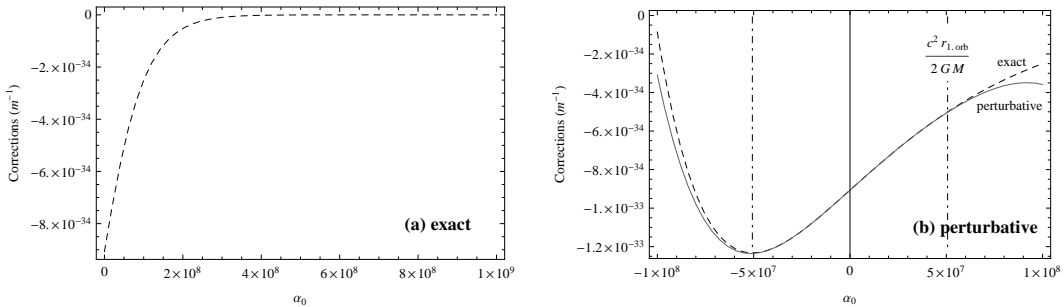


Figure 1: Plot of the exact (dashed line) and perturbative (continuous line) expressions for the corrections to the orbital differential equation (11) on the ELA metric background as a function of the parameter α_0 for Earth's orbit. The perturbative regime is valid for $|\alpha_0| < \alpha_{0,\text{max.pert}} \approx 5 \times 10^7$ (15):

- (a) plot of the exact expressions for $\alpha_0 > 0$, the corrections asymptotically vanish for large $\alpha_0 \gg \alpha_{0,\text{max.pert}}$;
- (b) plot of the exact and perturbative expressions for $\alpha_0 \in]-10^8, 10^8[$, the perturbative and exact expressions approximately match up to $|\alpha_0| = \alpha_{0,\text{max.pert}}$.

In addition, with respect to the bound (12), we note that it is also obeyed as long as the bound (15) is obeyed, hence for larger values of the coefficient α_0 the terms of order H_0^2 explicitly depending

on the time coordinate become relevant and must be included in the Lagrangian (8). For these cases there is no constant of motion directly associated with energy conservation. Specifically, conserved energy, would be given by the constant $E = \int d\tau (\delta\mathcal{L}/\delta\dot{t} - d/d\tau(\delta\mathcal{L}/\delta\dot{t}))$, such that, due to the complexity of the full equations of motion, it would be preferable to consider a numerical analysis to compute orbital motion.

We are proceeding assuming that the upper bound (15) is obeyed. To solve the differential equation (13) we start by solving the differential equation considering only the dominant term in the right-hand side of (13), hence obtaining [21,22]

$$u''_{0.H^2} + A u_{0.H^2} = \frac{GM}{J^2} B \Rightarrow u_{0.H^2} = \frac{1 + e \cos(\sqrt{A} \varphi)}{d}, \quad (16)$$

where e is the elliptical orbit eccentricity and d is defined in terms of the ellipse semi-major axis $r_{1.\text{orb}}$ as

$$d = r_{1.\text{orb}}(1 - e^2). \quad (17)$$

The standard General Relativity angular momentum J_0 and the angular momentum J are expressed in terms of the parameter d as

$$\begin{aligned} J_0 &= -\sqrt{GMd}, \\ J &= -\sqrt{GMd \frac{B}{A}} \approx \frac{1}{2} J_0 (\delta_B - \delta_A)_{J=J_0}, \end{aligned} \quad (18)$$

where to evaluate A and B , we have approximated the angular momentum by the respective Keplerian quantity, $J \approx J_0$.

Next let us compute the corrections to the solution $u_{0.H^2}$ by considering the remaining terms in the right-hand side of the differential equation (13) evaluated for the function $u_{0.H^2}$ (16) such that the full solution is

$$u = u_{0.H^2} + u_{\text{GR}.H^2} + u_{H^2}. \quad (19)$$

Here the functions $u_{\text{GR}.H^2}$ and u_{H^2} correspond respectively to the corrections to the Keplerian orbit's solution due to the Schwarzschild background and due to the ELA metric background approximated to order H_0^2 being, respectively, the solutions of the following differential equations

$$\begin{aligned} u''_{\text{GR}.H^2} + A u_{\text{GR}.H^2} &= \frac{3GM}{c^2} C u_{0.H^2}^2 = \frac{3GM}{c^2} C \frac{(1 + e \cos(\sqrt{A} \varphi))^2}{d^2}, \\ u''_{H^2} + A u_{H^2} &= -\left(\frac{H_0}{J}\right)^2 \frac{1}{u_{0.H^2}^3} + \alpha_0 \left(\frac{H_0}{J}\right)^2 \frac{GM}{c^2} \frac{1}{u_{0.H^2}^2} \\ &= -\left(\frac{H_0}{J}\right)^2 \frac{1}{(1 + e \cos(\sqrt{A} \varphi))^3} \\ &\quad + \alpha_0 \left(\frac{H_0}{J}\right)^2 \frac{GM}{c^2} \frac{1}{(1 + e \cos(\sqrt{A} \varphi))^2}, \end{aligned} \quad (20)$$

such that we obtain

$$\begin{aligned}
u_{\text{GR},H^2} &= \frac{C}{A} \frac{\alpha_{\text{GR}}}{d} \left(\left(1 + \frac{e^2}{2}\right) - \frac{e^2}{6} \cos(2\sqrt{A}\varphi) + \sqrt{A} e \varphi \sin(\sqrt{A}\varphi) \right), \\
u_{H^2}(\varphi) &= \frac{d^3 H_0^2}{A J^2 (1-e^2)} \left(\frac{\alpha_0 GM}{c^2 d} + \frac{(-4 + e^2) + 3e^2 \cos(2\sqrt{A}\varphi)}{4(1-e^2)(1+e \cos(\sqrt{A}\varphi))} \right. \\
&\quad \left. - \left(\frac{3}{2(1-e^2)} - \frac{\alpha_0 GM}{c^2 d} \right) \frac{2e \arctan \left(\sqrt{\frac{1-e}{1+e}} \tan \left(\frac{\sqrt{A}\varphi}{2} \right) \right) \sin(\sqrt{A}\varphi)}{\sqrt{1-e^2}} \right),
\end{aligned} \tag{21}$$

where

$$\alpha_{\text{GR}} = \frac{3GM}{c^2 d}. \tag{22}$$

Both the solutions u_{GR,H^2} and u_{H^2} have the same structure of the standard solution for Schwarzschild backgrounds [21, 22], the first term is a constant that can be neglected, the second term has a period that is a multiple of the period of solution u_{0,H^2} (16) contributing a small correction to the orbital period and the last term monotonically grows with increasing φ contributing to the orbital precession. This last result is justified by noting that the analytic continuation of the inverse of a function corresponds to the argument of the function (in this way $\arctan(\tan \varphi) = \varphi$ increases monotonically with φ). Due to the corrections to the Keplerian orbit's solution being small when compared to the dominant term, we can expand the trigonometric functions to lower order [21, 22]: $\cos(\sqrt{A}\varphi) = 1 - A^2\varphi^2/2 + O(\varphi^4)$, $\varphi \sin(\sqrt{A}\varphi) = 2\sqrt{A}\varphi^2/2 + O(\varphi^4)$ and $\arctan \left(\sqrt{(1-e)/(1+e)} \tan \left(\sqrt{A}\varphi/2 \right) \right) = A\sqrt{(1-e)/(1+e)}\varphi^2/2 + O(\varphi^4)$. Hence, neglecting the constant terms in the solutions u_{GR,H^2} and u_{H^2} (21) and gathering the several terms and respective coefficients for these lower order expansions, we obtain the full solution u (19)

$$\begin{aligned}
u &\approx \frac{1}{d} \left(1 + e \cos \left(\left(1 - \frac{\Delta\varphi_{\text{GR}}}{2\pi} - \frac{\Delta\varphi_{H^2}}{2\pi} \right) \varphi \right) \right) + u_{\text{osc,GR}} + u_{\text{osc},H^2}, \\
\frac{\Delta\varphi_{\text{GR}}}{2\pi} &= \alpha_{\text{GR}} \\
\frac{\Delta\varphi_{H^2}}{2\pi} &= -\frac{\delta_A}{2} + \alpha_{\text{GR}} \delta_C \left(1 + \frac{2e^2}{3d} \right) \\
&\quad + \frac{d^3 H_0^2}{(1-e)(1+e)^{\frac{3}{2}}} \left(\frac{\alpha_0}{c^2 d} - \frac{3}{2(1-e^2)GM} \right) + O(H_0^4), \\
u_{\text{osc,GR}} &= -\frac{\alpha_{\text{GR}}}{6d} e^2 \cos(2\varphi), \\
u_{\text{osc},H^2} &= \frac{(\delta_C - \delta_A)\alpha_{\text{GR}}}{6} e^2 \cos(2\varphi) + (H_0 d)^2 \frac{-4 + e^2 + 3e^2 \cos(2\varphi)}{4(1-e^2)^2 GM (1+e \cos \varphi)} + O(H_0^4)
\end{aligned} \tag{23}$$

where $\Delta\varphi_{\text{GR}}/(2\pi)$ is the standard precession per turn of the orbit due to General Relativity corrections on Schwarzschild backgrounds and $\Delta\varphi_{H^2}/(2\pi)$ is the precession per turn of the orbit due to the ELA metric background. As for the factor $u_{\text{osc,GR}}$ it is the General Relativity oscillatory factor correction to the orbit solution obtained for Schwarzschild backgrounds and u_{osc,H^2} is the oscillatory factor correction due to the ELA metric background.

To compute the observable period correction to the orbits due to the ELA metric background it is enough to consider the definition of the constant of motion $J d\tau = -d\varphi/u^2$ (9) and integrate

the infinitesimal proper time displacement $d\tau$ over one turn of the orbit such that we obtain

$$T = -\frac{1}{J} \int_0^{2\pi} \frac{d\varphi}{u^2} \approx -\frac{1}{J} \int_0^{2\pi} d\varphi \frac{1}{u_0^2} \left(1 - \frac{2u_{\text{osc.GR}}}{u_0} - \frac{2u_{\text{osc.H}^2}}{u_0} \right). \quad (24)$$

To directly compare the General Relativity corrections to the orbital period on the ELA metric background with the Keplerian orbital period and the General Relativity corrections on Schwarzschild backgrounds this integral can be factorized into the 3 components

$$\begin{aligned} T &= T_0 + \Delta T_{\text{GR}} + \Delta T_{H^2}, \\ T_0 &= -\frac{1}{J_0} \int_0^{2\pi} d\varphi \frac{1}{u_0^2} = \frac{2\pi r_{1,\text{orb}}^{\frac{3}{2}}}{\sqrt{GM}}, \\ \Delta T_{\text{GR}} &= +\frac{2}{J_0} \int_0^{2\pi} d\varphi \frac{u_{\text{osc.GR}}}{u_0^3} = -\frac{3\pi \sqrt{GM} r_{1,\text{orb}}^{\frac{1}{2}} e^4}{c^2(1-e^2)^2}, \\ \Delta T_{H^2} &= -\frac{1}{2} (T_0 + \Delta T_{\text{GR}}) (\delta_A - \delta_B) - \frac{2}{|J|} \int_0^{2\pi} d\varphi \frac{u_{\text{osc.H}^2}}{u_0^3} \\ &\approx -(\delta_A - \delta_B) \left(\frac{2\pi r_{1,\text{orb}}^{\frac{3}{2}}}{\sqrt{GM}} - \frac{3\pi \sqrt{GM} r_{1,\text{orb}}^{\frac{1}{2}} e^4}{2c^2(1-e^2)^2} \right) \\ &\quad + \frac{3\pi (\delta_A - \delta_C) r_{1,\text{orb}}^{\frac{3}{2}} e^4 \sqrt{GM}}{c^2(1-e^2)} + \frac{\pi r_{1,\text{orb}}^{\frac{9}{2}} (4+9e^2) H_0^2}{(GM)^{\frac{3}{2}}} + O(H_0^4), \end{aligned} \quad (25)$$

where T_0 is the classical Keplerian orbit period corresponding to the solution u_0 (7), ΔT_{GR} is the standard General Relativity period correction on Schwarzschild backgrounds corresponding to solution $u_{\text{osc.GR}}$ (23) and ΔT_{H^2} is the General Relativity period correction on the ELA metric background corresponding to solution $u_{\text{osc.H}^2}$ (23).

Planet	$\Delta T_{\text{GR}} \text{ (s/yr}^{-1}\text{)}$	$\frac{\Delta\varphi_{\text{GR}}}{2\pi} \text{ (arcsec/century}^{-1}\text{)}$
Mercury	$+2.35 \times 10^{-3}$	10.35
Venus	$+1.36 \times 10^{-9}$	5.30
Earth	$+3.64 \times 10^{-8}$	3.84
Mars	$+2.38 \times 10^{-5}$	2.54
Jupiter	$+4.95 \times 10^{-7}$	0.74
Saturn	$+4.24 \times 10^{-7}$	0.40
Uranus	$+1.21 \times 10^{-7}$	0.20
Neptune	$+8.44 \times 10^{-11}$	0.13
Pluto	$+5.15 \times 10^{-5}$	0.10

Table 2: Standard General Relativity corrections to the orbital period ΔT_{GR} (25) and orbital precession $\Delta\varphi_{\text{GR}}/2\pi$ (23) for each planet in the Solar System. These corrections are computed for the Solar Schwarzschild background with respect to the respective quantities for Keplerian orbits.

The values for the standard General Relativity corrections to the orbital period and orbital precession on Schwarzschild backgrounds are listed in table 2. These corrections correspond to orbital

motion on Schwarzschild backgrounds being well known and already accounted for in the model employed in the numerical analysis of the Solar System dynamics.

In addition, although on Schwarzschild backgrounds the orbital radius is not varying over time, on expanding backgrounds such as the ones described by the ELA metric, it is expected that the radius does vary as the background expands. The analytical solutions computed so far do not allow to estimate such variation for the orbital radius as we have approximated the ELA metric background by a static background with fixed Hubble rate $H = H_0$ (8). In the next section, from conservation of angular momentum, we estimate the orbital radius variation by considering approximately circular orbits.

3 Circular Orbits Approximation: Time Varying Orbital Radius

In this section, with the objective of estimating the orbital radius variation on backgrounds described by the ELA metric, we are analysing circular orbits on such backgrounds. In the non-relativistic velocity limit and for relatively small values of the radial coordinate ($r_1 \ll l_H = c/H$) the radial acceleration is

$$\ddot{r}_1 \approx -c^2 \Gamma^1_{00} \approx -\frac{GM}{r_1^2} + \frac{2(GM)^2}{c^2 r_1^3} + F_{H^2} + O(r_1^2 H^4) , \quad (26)$$

where in the right hand side of the equation, the first term is the usual classical Newton gravitational acceleration, the second term is the standard General Relativity correction on Schwarzschild backgrounds and the third term is the General Relativity correction of order H^2 for backgrounds described by the ELA metric (4)

$$F_{H^2} = +r_1 \left(1 - \frac{2GM}{c^2 r_1} \right)^{\alpha_0} \left(1 - \frac{(1 - \alpha_0)GM}{c^2 r_1} - (1 + q) \left(1 - \frac{2GM}{c^2 r_1} \right)^{\frac{1}{2} - \frac{\alpha_0}{2}} \right) H^2 . \quad (27)$$

To derive the orbital velocity let us consider the constant of motion corresponding to conservation of angular momentum, $J = -r_1^2 d\varphi/d\tau$. Particularizing to circular orbits for which the orbital velocity is constant, $v_{\text{orb}} = \sqrt{-r_1 \ddot{r}_1}$, and considering the usual definition of angular velocity $\dot{\varphi} = \omega = v_{\text{orb}}/r_1$ we obtain the following definition for the angular momentum $J_{\text{circ}}^2 = -\gamma^2 r_1^3 \ddot{r}_1|_{r_1=r_{1,\text{orb}}}$ such that for an orbit of radius $r_{1,\text{orb}}$ we obtain

$$J_{\text{circ}}^2 \approx GM r_{1,\text{orb}} \left(1 + \left(\frac{H r_{1,\text{orb}}}{c} \right)^2 \left(1 - \frac{2GM}{c^2 r_{1,\text{orb}}} \right)^{\alpha_0} \right) - F_{H^2} r_{1,\text{orb}}^3 . \quad (28)$$

Here $\gamma = dt/d\tau$ is the relativistic factor for the ELA metric (4) and dotted quantities represent derivation with respect to the coordinate time t . Specifically, in the limit of non-relativistic velocity $\dot{x}^\mu \ll c$, it is

$$\frac{d^2 r_1}{d\tau^2} \approx \gamma^2 \ddot{r}_1 \approx \frac{\ddot{r}_1}{1 - \frac{2GM}{c^2 r_1} - \left(\frac{H r_1}{c} \right)^2 \left(1 - \frac{2GM}{c^2 r_1} \right)^{\alpha_0}} . \quad (29)$$

For circular orbits, the main effect obtained due to the corrections on the ELA metric background correspond to a time varying radius. Such effect can be verified from conservation of angular momentum. To lowest order in time, H is expressed as $H(t) \approx H_0 - q_0 H_0^2 t$ such that assuming a non-varying Gravitational constant $\dot{G} = 0$ and non-varying mass $\dot{M} = 0$ we are left with the only possibility of a time-varying orbital radius $\dot{r}_{1,\text{orb}} \neq 0$. Hence differentiating equation (28) and

solving the equation $\dot{J}_{\text{circ}} = 0$ for $\dot{r}_{1,\text{orb}}$ we obtain, to lowest order in H_0 , the time dependence of the orbital radius

$$\begin{aligned} \frac{\dot{r}_{1,\text{orb}}}{r_{1,\text{orb}}} &\approx \frac{2q_0 (H_0 r_{1,\text{orb}})^3}{GM} \left(1 - \frac{2GM}{c^2 r_{1,\text{orb}}}\right)^{\frac{\alpha_0}{2} + \frac{1}{2}} \times \\ &\times \left(1 + q_0 - \left(1 - \frac{(2 - \alpha_0)GM}{c^2 r_{1,\text{orb}}}\right) \left(1 - \frac{2GM}{c^2 r_{1,\text{orb}}}\right)^{\frac{\alpha_0}{2} - \frac{1}{2}}\right) + O(H_0^5). \end{aligned} \quad (30)$$

As expected from cosmological expansion this expression increases with the orbital radius $r_{1,\text{orb}}$ and decreases with the mass M . Consistently at very large radius ($r_1 \sim l_H = c/H$) the gravitational potential is negligible ($1/r_1 \sim 0$) such that pure cosmological expansion is asymptotically recovered and no stable orbits exist ($\dot{r}_1/r_1 \sim 2q_0^2 H^3 r_1^3 > 0$).

As for the specific dependence of the orbital radius variation on the parameter α_0 it is positive for small values of $\alpha_0 \sim 0$ being of the same order of magnitude of the pure expansion effects, for growing positive values of this parameter, the radius variation decreases having a negative minimum value and then asymptotically vanishing in the limit $\alpha_0 \rightarrow +\infty$. This is actually expected, we note that in this limit the shift function is null, $\lim_{\alpha_0 \rightarrow +\infty} (1 - U_{\text{SC}}) = 0$, such that we exactly recover the SC metric, hence a Ricci flat space-time for which \dot{r}_1/r_1 is exactly null for all orbits. As for growing negative values of this parameter the radius variation increases up to a maximum positive value and then decreases monotonically. For large negative values of this parameter $\alpha_0 \ll 0$ the corrections with respect to Schwarzschild backgrounds become significantly higher with $\dot{r}_1/r_1 < 0$ being unbounded from below. As an example of the typical values of $\dot{r}_{1,\text{orb}}/r_{1,\text{orb}}$ as a function of the parameter α_0 are plotted in figure 2 the values of $\dot{r}_{1,\text{orb}}/r_{1,\text{orb}}$ for the Earth-Moon orbit and for Sun-Venus, Sun-Earth and Sun-Mars orbits.

We further note that the estimative for the orbital radius variation just computed is a valid approximation for elliptical orbits of small eccentricity $e \ll 1$, hence a fairly good approximation for all planetary orbits in the Solar system except for Mercury's and Pluto's orbits for which $e \sim 0.2$ such that these estimative correspond at most to a rough approximation to the orbital radius variation for both these planets.

Next, considering the General Relativity corrections on backgrounds described by the ELA metric, we will show that it is possible to map these corrections to the heuristic variation of the standard of measurement discussed in the introduction (either the variation of the AU or the variation of G), by matching the corrections to Kepler's third law on such backgrounds to the fitted value of the parameter \dot{AU} .

4 Variation of the Astronomical Unit

As already discussed in the introduction the heuristic variation of the AU (or equivalently the variation of the Gravitational constant G) obtained from numerical analysis of the Solar System dynamics indicates that some unmodelled gravitational interaction is present.

Here we are employing the ELA metric background to model such corrections to gravitational interactions. From the definition of the AU (1) and considering a non-varying Gravitational constant $\dot{G} = 0$ the effects that can match the heuristic variation of the AU are the decrease of the Sun's mass, the orbital period corrections and the orbital radius variation. The central mass reduction has the effect of increasing both the orbital radius and the orbital period such that angular momentum is conserved [4] and the equality (1) is maintained. Hence it is mapped to

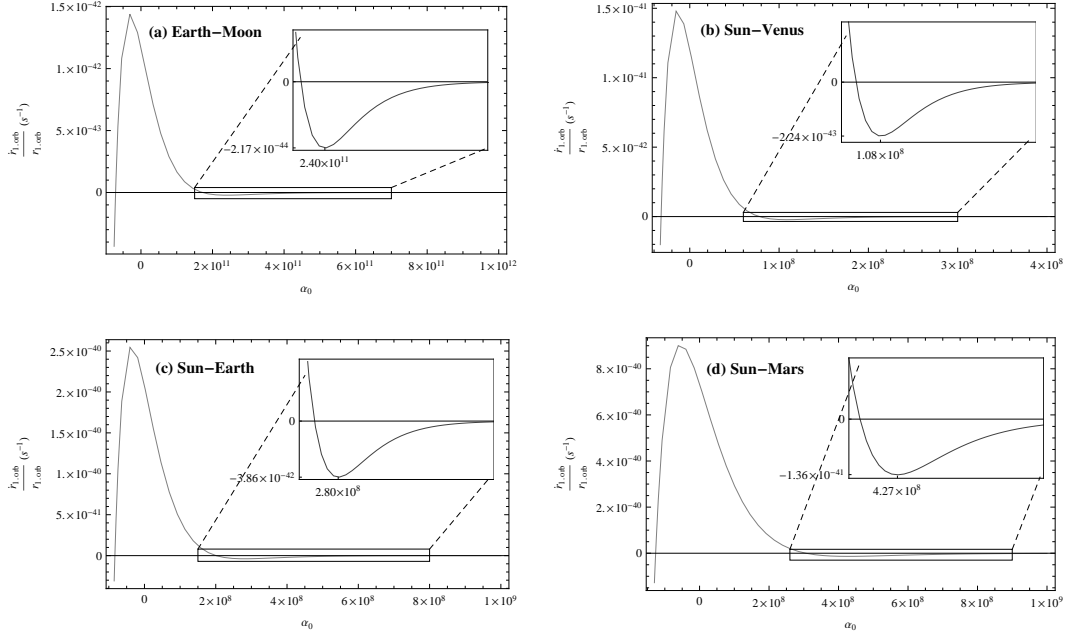


Figure 2: Examples of the profiles of the time variation rate of the orbital radius $\dot{r}_{1,orb}/r_{1,orb}$ (30) as a function of the parameter α_0 assuming the circular orbits approximation:

(a) for the Earth-Moon orbit, the maximum positive variation is $\dot{r}_{1,orb}/r_{1,orb} = 1.440 \times 10^{-42} s^{-1}$ corresponding to $\alpha_0 = -3.084 \times 10^{10}$ and the minimum negative variation for positive α_0 is $\dot{r}_{1,orb}/r_{1,orb} = -2.173 \times 10^{-44} s^{-1}$ corresponding to $\alpha_0 = 2.396 \times 10^{11}$;

(b) for the Sun-Venus orbit, the maximum positive variation is $\dot{r}_{1,orb}/r_{1,orb} = 1.483 \times 10^{-41} s^{-1}$ corresponding to $\alpha_0 = -1.397 \times 10^7$ and the minimum negative variation for positive α_0 is $\dot{r}_{1,orb}/r_{1,orb} = -2.238 \times 10^{-43} s^{-1}$ corresponding to $\alpha_0 = 1.085 \times 10^8$;

(c) for the Sun-Earth orbit, the maximum positive variation is $\dot{r}_{1,orb}/r_{1,orb} = 2.557 \times 10^{-40} s^{-1}$ corresponding to $\alpha_0 = -3.609 \times 10^7$ and the minimum negative variation for positive α_0 is $\dot{r}_{1,orb}/r_{1,orb} = -3.859 \times 10^{-42} s^{-1}$ corresponding to $\alpha_0 = 2.803 \times 10^8$;

(d) for the Sun-Mars orbit, the maximum positive variation is $\dot{r}_{1,orb}/r_{1,orb} = 9.043 \times 10^{-40} s^{-1}$ corresponding to $\alpha_0 = -5.498 \times 10^7$ and the minimum negative variation for positive α_0 is $\dot{r}_{1,orb}/r_{1,orb} = -1.365 \times 10^{-41} s^{-1}$ corresponding to $\alpha_0 = 4.271 \times 10^8$.

the variation of the AU as a positive contribution approximately given by $\dot{AU}/AU \approx -\dot{M}_\odot/M_\odot$. The estimative for the Sun's mass variation due to radiation and matter emission is (see [4, 7] and references therein)

$$\frac{\dot{M}_\odot}{M_\odot} = -6.7_{-3.1}^{+3.1} \times 10^{-14} (yr^{-1}) , \quad (31)$$

hence, from this estimative we obtain a equivalent contribution for the variation of the AU of

$$\Delta \dot{AU}_{1AU.\dot{M}} = -\frac{\dot{M}_\odot}{M_\odot} AU \approx +1.002_{-0.464}^{+0.464} (cm yr^{-1}) , \quad (32)$$

with AU evaluated in centimetre. As for the period correction ΔT_{H2} , due to General Relativity corrections on backgrounds described by the ELA metric, it is mapped to a correction to the

variation of the AU given by the following difference

$$\begin{aligned}\Delta \dot{A}U_{1AU.H^2} &= \frac{1}{1 \text{ yr}} \left(\left(G M_\odot \left(\frac{T_{1AU.0} + \Delta T_{1AU.GR} + \Delta T_{1AU.H^2}}{2\pi} \right)^2 \right)^{\frac{1}{3}} \right. \\ &\quad \left. - \left(G M_\odot \left(\frac{T_{1AU.0} + \Delta T_{1AU.GR}}{2\pi} \right)^2 \right)^{\frac{1}{3}} \right) \\ &\approx \frac{2}{3} \frac{1}{1 \text{ yr}} \frac{\Delta T_{1AU.H^2}}{T_{1AU.0}} AU \text{ (cm yr}^{-1}\text{)},\end{aligned}\quad (33)$$

with AU evaluated in centimetre and $T_{1AU.0}$ and $\Delta T_{1AU.H^2}$ corresponding to the period and the period variation over one revolution of the orbit evaluated both in the same units of time (e.g. second). The factor of yr^{-1} is explicitly written converting between the time units for $T_{1AU.0}$ and years such that 1 yr corresponds to one revolution of a Keplerian orbit with semi-major axis of length of $1 AU$. We remark that one Keplerian Earth year differs from $T_{1AU.0}$ by 5.21 s . In addition, a non-null orbital radius variation, is directly mapped to the variation of the AU such that for an orbital radius of $r_{1.\text{orb}.1AU} = 1 AU$ we obtain

$$\Delta \dot{A}U_{1AU.\dot{r}} = \frac{\dot{r}_{1.\text{orb}.1AU}}{r_{1.\text{orb}.1AU}} \frac{T_{1AU.0}}{1 \text{ yr}} AU \text{ (cm yr}^{-1}\text{)}, \quad (34)$$

where AU is evaluated in centimetre and $\dot{r}_{1.\text{orb}.1AU}/r_{1.\text{orb}.1AU}$ and $T_{1AU.0}$ are evaluated in the same units of time (e.g. second).

Both the contributions (33) and (34) depend on the value of the metric parameter α_0 . To obtain a precise estimative for the values of these contributions it would be required a numerical analysis of the Solar System dynamics, however to further proceed analytically let us consider the approximate analytical estimative discussed in the previous sections. Specifically, the contribution (33) is evaluated by considering the period correction (25) computed for static analytical orbital solutions and, the contribution (34), is evaluated by considering the orbital radius variation (30) computed for circular orbital solutions. These estimative are a fairly good approximation except for the orbital radius variation for the planet Mercury and the planet Pluto due to the relatively high eccentricity of their orbits. Nevertheless, as we are going to show next, the main contributions that are mapped to the variation of the AU are due to the decrease of the Sun's mass (32) and to the General Relativity corrections to the orbital period on the ELA metric background (33) such that the contribution due to the orbital radius variation (34) is negligible being lower than the remaining contributions by a factor of 10^{-19} at Mercury's orbit and by a factor of 10^{-10} at Pluto's orbit, hence being negligible for the estimative obtained. For completeness of the analytical expressions we are keeping these contributions in the following derivations.

We are proceeding to evaluate an analytical estimative for the variation of the AU by considering the orbits of the nine planets in the Solar System. We note that for each orbit, the corrections to Kepler's third law are proportional to the orbital radius $r_{1.\text{orb}}$. To relate these corrections to the variation of the AU it is required to rescale the contributions from each planetary orbit to the AU length, hence to multiply each contribution by the factor $AU/r_{1.\text{orb}}$. Specifically, for a given orbit of semi-major axis $r_{1.\text{orb}}$, the respective mapped variation of the AU, for an observer at the Sun, is

$$\dot{A}U_{\text{orbit}} = \left(-\frac{\dot{M}_\odot}{M_\odot} + \frac{\dot{r}_{1.\text{orb}}}{r_{1.\text{orb}}} \frac{T_{1AU.0}}{1 \text{ yr}} + \frac{2}{3} \frac{1}{\text{yr}} \frac{\Delta T_{H^2}}{T_{\text{orb}.0}} \frac{T_{1AU.0}}{T_{\text{orb}.0}} \right) AU, \quad (35)$$

where $T_{\text{orb}.0}$ and ΔT_{orb} are, respectively, the Keplerian orbital period and the General Relativity correction to the orbital period of each planetary orbit on the ELA metric background with respect

to Schwarzschild backgrounds. $T_{1AU,0}$ is the Keplerian period corresponding to the definition of the AU (1) and in the last term the factor $T_{1AU,0}/T_{orb,0}$ scales the temporal variation from each planet year to the AU year. For a given planetary orbit these corrections have the effect of a significant decrease on the variation of the AU for large positive values of $\alpha_0 \gg 0$ ($\dot{AU} < 0$) and of a significant increase of the AU for large negative values of $\alpha_0 \ll 0$ ($\dot{AU} > 0$). The dependence of the parameter \dot{AU}_{orbit} on the values of α_0 for the inner planets of the Solar System, Mercury, Venus, Earth and Mars is plotted in figure 3.

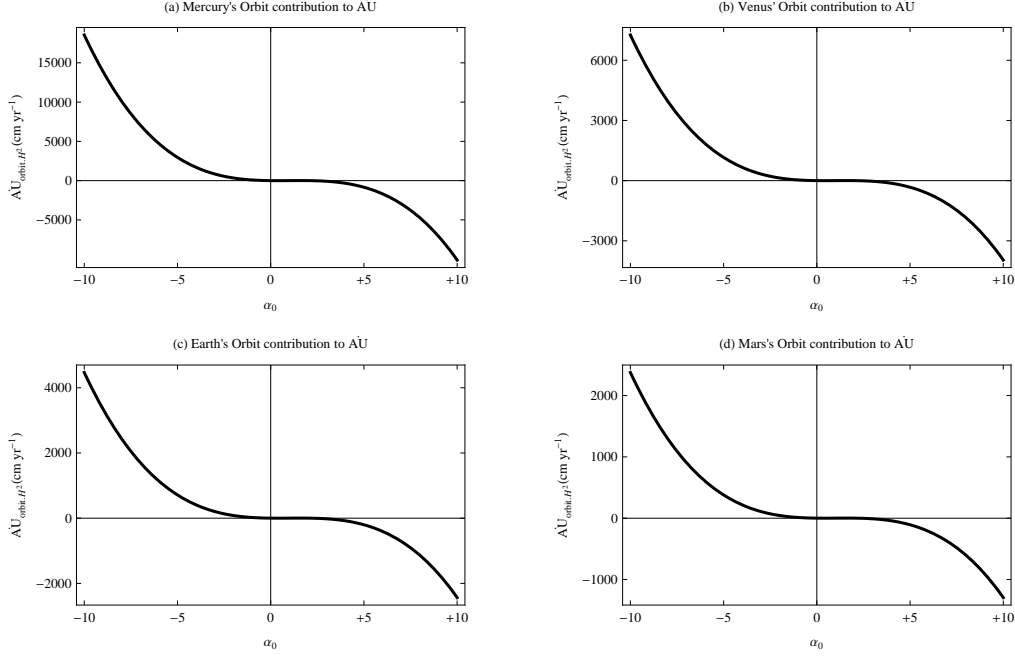


Figure 3: Corrections \dot{AU}_{orbit} (35) to the variation of the AU as a function of the constant metric parameter α_0 for (a) the orbit of Mercury; (b) the orbit of Venus; (c) the orbit of Earth; (d) the orbit of Mars.

As for the corrections to Kepler's third law as perceived for Earth based range measurements it is computed by evaluating the difference between the corrections corresponding to the geodesic motion of Earth and the corrections corresponding to the geodesic motion of the planet for which the range measurement is being considered. Hence, for range measurements between Earth and any other planet in the Solar System, these corrections are mapped to a variation of the AU by

$$\dot{AU}_{range} = \left(r_{1,orb.Earth} \dot{AU}_{orbit.Earth} - r_{1,orb.planet} \dot{AU}_{orbit.planet} \right) \frac{1}{\langle \Delta r_{1,Earth-planet} \rangle}, \quad (36)$$

where $\dot{AU}_{orbit.Earth}$ and $\dot{AU}_{orbit.planet}$ correspond to \dot{AU}_{orbit} (35) evaluated for the orbit of Earth and the orbit of each planet, respectively. The distance $\langle \Delta r_{1,Earth-planet} \rangle$ corresponds to the average distance between Earth and the planet assuming that both have Keplerian orbits. Planetary orbits of distinct bodies in the Solar System are generally not locked to each other, however to compute an estimative to this average value we are parameterizing Earth's orbit by the angular parameter $T_{planet} \varphi / (1 s)$ and the planet orbit, for which the range measurement is being considered, by $T_{Earth} \varphi / (1 s)$ with $\varphi \in [0, 2\pi]$ such that the average is performed over $\mathcal{T} = T_{planet} / (1 s)$ periods of the Earth orbit and $\mathcal{T}_E = T_{Earth} / (1 s)$ periods of the planet orbit. Further aligning both the Earth and planet perihelion with the x axis, considering a rotation of the planet orbit

around the x axis by its orbital inclination angle δ_{orb} followed by a rotation around the z axis by the angle between the longitudes of the planet's perihelion and Earth's perihelion $\delta_{per} - \delta_{per.E}$ (given in table 1), we obtain the following Keplerian estimative for the average distance between Earth and any other planet in the Solar System

$$\begin{aligned}
\langle \Delta r_{1.\text{Earth-planet}} \rangle &= \frac{1}{2\pi} \int_0^{2\pi} d\varphi \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2} , \\
\Delta x &= \frac{\cos(\mathcal{T} \varphi)}{u_{0.E}(\mathcal{T} \varphi)} - \frac{1}{u_0(\mathcal{T}_E \varphi)} \left(\cos(\mathcal{T}_E \varphi) \cos(\delta_{per} - \delta_{per.E}) , \right. \\
&\quad \left. + \sin(\mathcal{T}_E \varphi) \cos(\delta_{orb}) \sin(\delta_{per} - \delta_{per.E}) \right) \\
\Delta y &= \frac{\sin(\mathcal{T} \varphi)}{u_{0.E}(\mathcal{T} \varphi)} - \frac{1}{u_0(\mathcal{T}_E \varphi)} \left(\cos(\mathcal{T}_E \varphi) \sin(\delta_{per} - \delta_{per.E}) , \right. \\
&\quad \left. - \sin(\mathcal{T}_E \varphi) \cos(\delta_{orb}) \cos(\delta_{per} - \delta_{per.E}) \right) \\
\Delta z &= \frac{\sin(\mathcal{T}_E \varphi) \sin(\delta_{orb})}{u_0(\mathcal{T}_E \varphi)} ,
\end{aligned} \tag{37}$$

where the functions $u_{0.E}$ and u_0 correspond, respectively, to the inverse of the orbital radius for Earth Keplerian orbit and the planet Keplerian orbit (7).

Further noting that, when performing numerical analysis of the Solar System dynamics, the fitted variation of the AU is approximately a linear effect [4] being independent of each planet's mass, a estimative for the average value of $\dot{A}U_{\text{orbit}}$ (35) can be obtained by a simple average of the contributions due to each planetary orbit

$$\langle \dot{A}U \rangle_{\text{orbit}} = \frac{\sum_{i=1}^9 \dot{A}U_{\text{orbit}.i}}{9} . \tag{38}$$

As for the average value for the contribution of $\dot{A}U_{\text{range}}$ to the experimental data measurement can be estimated by an average weighted by the number of events N_i divided by the respective rms residuals σ_i

$$\langle \dot{A}U \rangle_{\text{range}} = \frac{\sum_{i=1, i \neq 3}^9 \dot{A}U_{\text{range}.i} \frac{N_i}{\sigma_i}}{\sum_{i=1, i \neq 3}^9 \frac{N_i}{\sigma_i}} . \tag{39}$$

In this expression the index i runs from 1 to 9 referring to the planets in the Solar System listed in table 1 and $\dot{A}U_{\text{orbit}.i}$ are the variation of the AU for each planetary orbit in the Solar System (35). The values of the weights $w_i = N_i/\sigma_i$ are computed from table 2 and table 3 of [4] being $\omega_1 = 4.65$, $\omega_2 = 35692.38$, $\omega_3 = 0$, $\omega_4 = 469279.08$, $\omega_5 = 74.86$, $\omega_6 = 348.02$, $\omega_7 = 62.40$, $\omega_8 = 64.66$ and $\omega_9 = 38.79$.

Hence, when considering the ELA metric background, there will be two distinct corrections which can be mapped into the heuristic fit of the parameter $\dot{A}U$. Let us recall that the numerical analysis of the Solar System dynamics has, generally, two distinct procedures [23]. First the ephemerides are built considering as the base model only the well established General Relativity

gravitation interactions on Schwarzschild backgrounds. Then the ephemerides are numerically integrated by considering a wide number of parameters which generally may include corrections to the gravitational interactions on Schwarzschild backgrounds, for example the PPN parameters γ , β and α , as well as the variation of the AU, corresponding to the parameter \dot{AU} or the variation of the Gravitational constant corresponding to the parameter \dot{G} . It is from this second numerical analysis that an heuristic fit to the parameter $\langle \dot{AU} \rangle_{Fit}$ is obtained (3). Therefore when considering the ELA metric background to model gravitational interactions corrections to the gravitational interactions on Schwarzschild backgrounds it is required to consider corrections to both of these procedures. In particular when mapping the modelled correction to an heuristic variation of the AU with respect to some fixed length AU_0 we obtain that the average value with respect to the ephemerides data is $\langle AU \rangle_{eph} = AU_0 + \langle \dot{AU} \rangle_{range}$ and, when fitting the ephemerides data to a heliocentric model of the Solar System, we obtain $\langle AU \rangle_{eph} = AU_0 + \langle \dot{AU} \rangle_{Fit} - \langle \dot{AU} \rangle_{orbit}$, where the contribution $\langle \dot{AU} \rangle_{Fit}$ corresponds to the heuristic fit (3). Hence matching these two expressions we obtain the following map between the heuristic fit to the parameter \dot{AU} (3) and the modelled average contributions (38) and (39)

$$\langle \dot{AU} \rangle_{Fit} = \langle \dot{AU} \rangle_{orbit} + \langle \dot{AU} \rangle_{range} . \quad (40)$$

Recalling that the ELA metric interpolates between local gravitational backgrounds and the expanding cosmological background we note that the previous discussion is actually consistent with the fact that expansion effects are proportional to the distance between the observer and the observed event, hence for range measurements it is required to consider the difference between geodesical motion of the observer (at Earth) and the geodesical motion of the remaining planets while for heliocentric orbital motion it is considered only the geodesic motion for each planet (the observer is at the Sun).

Generally we could consider a variation of the functional parameter α across the Solar System such that, for each planetary orbit, this metric parameter would be given by an approximate constant value $\alpha_{0,i}$. However an exact fit to ephemerides would require a full numerical analysis including the gravitational corrections due to the ELA metric background. Nevertheless, for analytical analysis purposes, let us simply consider an approximately constant coefficient α_0 across the Solar System. Hence, from the map (40) and for the value of the heuristic fit (3) we obtain

$$\langle \dot{AU} \rangle_{Fit} = (0.75 \pm 0.60) \text{ cm yr}^{-1} \Leftrightarrow \alpha_0 = 1.01^{+1.20}_{-1.22} . \quad (41)$$

We note that the dependence of \dot{AU} on α_0 has an inflexion point near $\alpha_0 = 1$ (see figure 2), the relatively large uncertainty on the value of α_0 is mainly due to the proximity to this inflexion point. In table 3 are listed the values of the contributions from each planetary orbit in the Solar System to the estimative (41). The correction to the orbital period for the value of the parameter α_0 is approximately constant for all planetary orbits

$$\Delta T_{H^2} \frac{1}{yr} \frac{T_{1AU.orb.0}}{T_{orb.0}} = -0.069^{+6.038}_{-6.038} \times 10^{-6} \text{ s yr}^{-1} , \quad (42)$$

where ΔT_{H^2} corresponds to the period correction per revolution for each planet (25). Although this correction is enough to map the heuristic fit to the variation of the AU, we note that it is negligible for most of other purposes, even for archaeological fits to the variation of the Solar System parameters we obtain at most a variation of the Earth year by $\pm 1.7 h$ over a period of 10^9 years, hence within the uncertainty of such estimative [7].

Planet	$\langle \Delta r_{1.\text{Earth}-\text{planet}} \rangle$ ($\times 10^{11} m$)	$\Delta \dot{A}U_{\text{orb}, \dot{r}}$ ($cm yr^{-1}$)	$\Delta \dot{A}U_{\text{orb}, H^2}$ ($cm yr^{-1}$)	$\dot{A}U_{\text{orbit}}$ ($cm yr^{-1}$)	$\dot{A}U_{\text{range}}$ (yr^{-1})
Mercury	1.552	5.55×10^{-21}	$-0.091^{+7.921}_{-7.921}$	$+1.093^{+7.921}_{-7.921}$	$0.580^{+1.117}_{-1.117}$
Venus	1.700	3.62×10^{-20}	$-0.036^{+3.101}_{-3.101}$	$+1.038^{+3.101}_{-3.101}$	$0.241^{+0.295}_{-0.295}$
Earth	—	9.57×10^{-20}	$-0.022^{+1.908}_{-1.908} \times 10^{-1}$	$+1.024^{+1.908}_{-1.908}$	—
Mars	2.525	3.38×10^{-19}	$-0.012^{+1.014}_{-1.014} \times 10^{-1}$	$+1.014^{+1.014}_{-1.014}$	$-0.309^{+0.215}_{-0.215}$
Jupiter	7.842	1.35×10^{-17}	$-0.018^{+1.607}_{-1.607} \times 10^{-1}$	$+1.004^{+0.161}_{-0.161}$	$-0.801^{+0.204}_{-0.204}$
Saturn	14.28	8.30×10^{-17}	$-0.074^{+6.478}_{-6.478} \times 10^{-2}$	$+1.003^{+0.065}_{-0.065}$	$-0.895^{+0.135}_{-0.135}$
Uranus	28.72	6.76×10^{-16}	$-0.026^{+2.269}_{-2.269} \times 10^{-2}$	$+1.005^{+0.023}_{-0.023}$	$-0.949^{+0.077}_{-0.077}$
Neptune	45.00	2.60×10^{-15}	$-0.013^{+1.157}_{-1.157} \times 10^{-2}$	$+1.002^{+0.012}_{-0.012}$	$-0.968^{+0.052}_{-0.052}$
Pluto	57.23	5.89×10^{-15}	$-0.089^{+7.690}_{-7.690} \times 10^{-3}$	$+1.002^{+0.008}_{-0.008}$	$-1.008^{+0.042}_{-0.042}$

Table 3: *Corrections to $\dot{A}U$ mapped from the corrections to Kepler’s third law on the ELA metric background with respect to Schwarzschild backgrounds. For each planet it is listed, in the first column the average distance between the planet and Earth for Keplerian orbits $\langle \Delta r_{1.\text{Earth}-\text{planet}} \rangle$ (37), in the second column the contribution due to the orbital radius variation (the second term in equation (35)) for which the uncertainty is at least 8 orders of magnitude below the quoted values, in the third column the contribution due to the period correction on the ELA metric background (the third term in equation (35)), in the fourth column the total correction to the variation of the AU for heliocentric orbital motion $\dot{A}U_{\text{orbit}}$ (35) and in the fifth column the total correction to the variation of the AU for Earth based range measurements $\dot{A}U_{\text{range}}$ (36).*

As for the values for the corrections to the orbital precession and orbital radius variation for each planet are listed in table 4. In particular the orbital precession corrections are lower by more than 10 orders of magnitude when compared to the respective values for Schwarzschild backgrounds and the orbital radius variation range from $\dot{r}_{1.\text{orb}}/r_{1.\text{orb}} \sim 10^{-32} \text{ century}^{-1}$ for mercury up to $\dot{r}_{1.\text{orb}}/r_{1.\text{orb}} \sim 10^{-26} \text{ century}^{-1}$, hence being well below any other estimative for these variations [7]. We recall that a variation of the AU does not imply an orbital radius variation [4–6] and further remark that the orbital radial variation obtained from numerical analysis of ephemerides in [4] are due to the variation of the Gravitational constant $\dot{G} \neq 0$, hence not directly comparable with the values obtained here, for which the quoted orbital radius variation in table 4 is due to the ELA metric background alone. In the framework discussed here there will be an extra contribution to the orbital radius variation due to the decrease of the Sun’s mass of order $10^{-12} \text{ century}^{-1}$ to $10^{-11} \text{ century}^{-1}$ similarly to the fit obtained in [4, 26].

With respect to the ELA metric background we note that the value of the metric parameter α_0 parameterizes the local anisotropic corrections with respect to the isotropic cosmological background, specifically space-time is locally isotropic for $\alpha_0 = 0$ which corresponds to the isotropic background described by the McVittie metric [15]. The value of $\alpha_0 \approx 1.01$ (41) corresponds to a relatively small perturbation to the isotropic background which, as has been shown, corresponds to relatively small corrections to the orbital parameters. Consistently with this discussion we remark that, when compared with a isotropic variation of the AU (or equivalently, to the variation of the Gravitational constant G), the corrections due to the ELA metric background to the orbital period are relatively more relevant than the corrections to the orbital precession and orbital radius variation. It is due to the background anisotropy between the radial direction and angular directions that such effect is attainable. In addition we recall that, when considering point-mass objects, as we approach the SC horizon the metric exponent α should be greater or equal to

Planet	$\frac{\Delta\varphi_{H^2}}{2\pi} (\text{arcsec century}^{-1})$	$\frac{\dot{r}_{1,\text{orb}}}{r_{1,\text{orb}}} (\text{century}^{-1})$
Mercury	-2.28×10^{-19}	3.71×10^{-32}
Venus	-5.82×10^{-19}	2.42×10^{-31}
Earth	-9.46×10^{-19}	6.40×10^{-31}
Mars	-1.78×10^{-18}	2.26×10^{-30}
Jupiter	-1.12×10^{-17}	9.01×10^{-29}
Saturn	-2.79×10^{-17}	5.55×10^{-28}
Uranus	-7.96×10^{-17}	4.52×10^{-27}
Neptune	-1.56×10^{-16}	1.74×10^{-26}
Pluto	-2.35×10^{-16}	3.94×10^{-26}

Table 4: *Corrections to precession and orbital radius variation due to the General Relativity corrections for the background described by the ELA metric. The estimative uncertainty for each of these values is several orders of magnitude below the quoted values.*

$\alpha(r_{1,\text{SC}}) = 3$ to ensure that space-time is singularity free at this horizon. This requirement is not absolutely necessary as the real Sun is not a point-like mass being instead an extended spheroid, hence without an event horizon. Nevertheless we further note that the uncertainty on the estimate of the constant α_0 is relatively large and we may expect that a functional parameter α varying across the Solar System would possible allow for a better fit to ephemerides, hence we may conjecture that its value should be decreasing with growing heliocentric distances being close to $\alpha = 3$ near the Sun. This discussion is not conclusive being required a numerical analysis of the Solar System dynamics including the corrections due to the ELA metric background to actually verify if such a profile for the values of α is the best fit to planetary motion.

5 Conclusions

In this work we have mapped the General Relativity corrections to Kepler's third law on backgrounds described by the ELA metric (4) to the heuristic variation of the AU estimated from numerical analysis of the Solar System dynamics on Schwarzschild backgrounds. These corrections together with the decrease of the Sun's mass by radiation emission fully account for the fitted value for the variation of the AU (3). Reflecting the anisotropic nature of the ELA metric background, the more relevant contribution to such modelling is due to the orbital period corrections, being the contributions from the orbital radius variation negligible.

The constant value for the metric parameter that matches the quoted variation of the AU is $\alpha_0 = +1.01^{+1.20}_{-1.22}$ (41), hence relatively close to the value $\alpha_0 = 0$ which corresponds to the isotropic background described by the McVittie metric. For completeness let us further note that other effects which may be relevant on backgrounds described by the ELA metric such as the corrections to the Doppler shift for range measurements are, for this range of values of the metric parameter negligible, being of the same order of magnitude of the effects attributed to isotropic cosmological expansion [20, 27]. Following the same arguments we further conclude that, within the Solar System the contribution to the cosmological mass-energy density within the Solar System due to the ELA metric background is negligible [28] (for further details see [20]).

Hence we have shown that the heuristic variation of the AU (or equivalently the heuristic variation of the Gravitational constant G) [2, 4] can alternatively be modelled by the ELA metric background parameterizing the corrections to gravitational interactions within the Solar System. For analytical analysis purposes we have considered a constant metric parameter α_0 for which we obtain a relatively high uncertainty. More generally a radially symmetric functional parameter with varying value across the Solar System would, in principle, allow for a significant reduction of such uncertainty. The results obtained here are enough to motivate a numerical analysis of the Solar System dynamics including the ELA metric background corrections to gravitational interactions. This framework solves the problem of the unwelcome variation of the measurement standard (whether the AU whether G) and constitutes a playground for testing the ELA metric background in the most well known of all the astrophysical systems, the Solar System. We leave such study to another work.

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